ON THE COMPLETE LOGARITHMIC SOLUTION OF THE CUBIC EQUATION.

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In this article we shall consider our equation to be in one of the two forms

$$(a)\ x^3 = qx + r \qquad (b)\ x^3 + qx = r$$

in which we may consider both r and q positive without loss of generality since changing the sign of r changes the signs of the roots.

The cubic and quadratic equations arising in practical work are, since their coefficients usually contain decimals, very different from the ideal ones of the classroom. The direct transformation of a complete cubic, all of whose constants are decimal fractions of say four places, by Horner's method into one of the above trinomial cubics is laborious. It is, however, possible easily to effect this transformation logarithmically by the use of tables of logarithms of sums and differences. Consider the general cubic

$$az^3 + bz^2 + cz + d = 0.$$

Setting

$$z = \frac{b}{3a}(x - 1), \quad Q = \frac{9ac}{b^2}, \quad R = \frac{27a^2d}{b^3},$$

we have

$$q = |(Q - 2) - 1|, \quad r = |(Q - 2) - R|,$$

and therefore

$$\log q = \log |(Q - 2) - 1|, \quad \log r = \log |(Q - 2) - R|,$$

considering q to be in (a) or (b) according as $Q - 3$ is negative or positive, and the roots to be $\pm x_1$, $\pm x_2$, $\pm x_3$ according as $Q - R - 2$ is positive or negative. Using a sum or difference table, according to the positive or negative qualities of the quantities involved, we first find $\log |Q - 2|$ and then $\log |(Q - 2\ - 1|$ and $\log |(Q - 2) - R|$. It is quicker to find $\log |(Q-2)-1|$ from the sum or difference table after having found $\log |Q-2|$ than it is to find $\log |Q - 3|$, hence the above arrangement appears to be the best. This solution may be justly called purely logarithmic, since only the logarithms of the constants are required throughout the computation, and no other than the ordinary tables are employed.

Corresponding to (a), we have the identities

$$\cos^3 \varphi = \tfrac{3}{4} \cos \varphi + \tfrac{1}{4} \cos 3\varphi$$

$$\cosh^3 \varphi = \tfrac{3}{4} \cosh \varphi + \tfrac{1}{4} \cosh 3\varphi$$

the former to be used when all the roots are real, the latter when there is but one real root.

Corresponding to (b), which has but one real root, we have

$$\sinh^3 \varphi + \tfrac{3}{4} \sinh \varphi = \tfrac{1}{4} \sinh 3\, \varphi.$$

A single formula suffices for the determination of the imaginary roots of both (a) and (b). Let the imaginary root of either (a) or (b) be $-\xi \pm \eta\sqrt{-1} = \mu(\cos\theta \pm \sqrt{-1}\sin\theta)$. If x_1 is the real root, we then have $\xi = \frac{1}{2}x_1$ and $r = x_1\mu^2$, or $\mu = \sqrt{r_1/x_1}$. Therefore

$$\cos\theta = \sqrt{\frac{x_1^3}{4r}}, \quad \eta = \sin\theta\cdot\sqrt{\frac{r}{x_1}}.$$

This is a much simpler trigonometric solution for the complex roots than that given by Chauvenet,* which is not only involved but also requires separate formulæ for (a) and (b). The latter solution is given for comparison.

For (a) put

$$\sin\theta = \sqrt{\frac{4q^3}{27r^2}}, \quad \tan\tfrac{1}{2}\varphi = \sqrt[3]{\tan\tfrac{1}{2}\theta},$$

then

$$\eta = \tfrac{1}{2}x_1\sqrt{3}\cdot\cos\varphi.$$

For (b) put

$$\tan\theta = \sqrt{\frac{4q^3}{27r^2}}, \quad \tan\tfrac{1}{2}\varphi = \sqrt[3]{\tan\tfrac{1}{2}\theta},$$

then

$$\eta = \tfrac{1}{2}x_1\sqrt{3}\cdot\sec\varphi.$$

The former trigonometric solution can be generalized into a logarithmic method of computing the two rema ning roots (either conjugate complex or real numbers) when $n - 2$ roots of an equation of the nth degree are known, circular functions being used when the remaining pair are imaginary and hyperbolic functions when they are real. Thus for computing the complex roots $-\xi \pm \eta\sqrt{-1}$ of

$$x^n + p_1x^{n-1} + \cdots + p_n = 0$$

when $n - 2$ of its roots are known, we have

$$\xi = \tfrac{1}{2}(p_1 + x_1 + \cdots + x_{n-2}), \quad \mu = \sqrt{\left|\frac{p_n}{x_1 \cdots x_{n-2}}\right|},$$

$$\cos\theta = \left|\frac{\xi}{\mu}\right|, \quad \eta = \mu\sin\theta.$$

* Plane and Spherical Trigonometry, pp. 97, 98.

If the remaining pair are real and are represented by $-\xi \pm \eta$, where and μ have their previous values we have

$$\cosh \theta = \left|\frac{\xi}{\mu}\right|, \quad \eta = \mu \sinh \theta.$$

This is particularly useful as a logarithmic solution of the quadratic equation, where more often than not in practical work the coefficients contain decimal fractions, and the algebraic formulæ are consequently rendered cumbersome for computation.

Substituting the numerical values of the logarithms of the constants, the above can be arranged in the following table.

Table for the Complete Logarithmic Solution of the Cubic Equation.

$$(a)\ x^3 = qx + r, \quad (b)\ x^3 + qx = r.$$

q and r are both positive. Changing the sign of r changes the signs of the roots.

I. Case (a). $4q^3 < 27r^2$. Three real roots x_1, x_2, x_3,

$$\log \cos 3\varphi = 0.41465 + \log r - \tfrac{3}{2} \log q, \tag{1}$$

$$\log x_1 = 0.06247 + \tfrac{1}{2} \log q + \log \cos \tfrac{1}{3}\theta, \tag{2}$$

$$\log (-x_2) = 0.06247 + \tfrac{1}{2} \log q + \log \cos \tfrac{1}{3}(\pi - \theta),$$

$$\log (-x_3) = 0.06247 + \tfrac{1}{2} \log q + \log \cos \tfrac{1}{3}(\pi + \theta),$$

where θ is the smallest positive value of 3φ satisfying (1).

II. Case (a). $4q^3 < 27r^2$. One real root x_1.

Replace cos by cosh in (1) and (2).

III. Case (b). One real root x_1.

Replace cos by sinh in (1) and (2).

IV. To calculate the imaginary roots in II and III.

Let them be $-\xi \pm \eta\sqrt{-1} = \mu(\cos\theta \pm \sqrt{-1}\sin\theta)$; then

$$\xi = \tfrac{1}{2}x_1,$$

$$\log \cos \theta = \tfrac{3}{2} \log x_1 + 9.69897 - r \tfrac{1}{2} \log - 10,$$

$$\log \eta = \tfrac{1}{2} (\log r - \log x_1) + \log \sin \theta.$$

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